**Characteristics roots and Characteristics vectors of Square Matrices**

**Definition of Characteristics roots:** Let A be any square matrix. The | A-xI | is said to be characteristic polynomial of the matrix A and the equation

| A-xI |=0

is said to be a characteristic equation of A, where x is any scalar.

The roots of the above equation is said to be an eigenvalues or characteristic roots or characteristic values of the matrix A.

**Definition of Characteristics vectors:** Let  be an eigenvalue of a square matrix A. Then a non-zero vector X is said to be an eigenvector or characteristic vector of the matrix A corresponding to the eigenvalue  if AX=X.

**Relation between eigenvalue and eigenvector:**

1.  is an eigenvalue of a square matrix A there exist a non-zero vector X such that AX=X.
2. If X is an eigenvector of a matrix A corresponding to the eigenvalue  , then kX is also an eigenvector of A corresponding to the eigenvalue  , where k is any non-zero scalar.

Hints: Since X is an eigenvector of the matrix A corresponding to the eigenvalue  ,

 AX=X  (1)

We now have,

 A(kX) =(Ak)X

 =(kA)X

=k(AX)

 =k(X) (by (1))

 =(k)X

 =(  k)X

 =(Kx)

This shows that kX is an eigenvector of A corresponding to the eigenvalue .

**Working rule for finding eigenvalue and eigenvector:**

Let  be a square matrix of order n. First we find the characteristic equation

 | A-xI |=0 of the matrix A. This equation will give us n-roots, which will represent eigenvalues of the matrix A. If  is any one of these eigenvalue of A, then the corresponding eigenvectors of A will be given by the non-zero vectors  such that AX=X.

 **Question:** If A is a square matrix, then show that A and  (transpose of A) has same eigenvalues and eigenvectors.

**Solution:** We have,

Characteristic polynomial of  =| - xI |

 =| |-| xI |

 =| A | - | xI |

 =| A-xI |

 = Characteristic polynomial of A.

Since the Characteristic polynomial of A and  are same, therefore their eigenvalues as well as eigenvectors are also same.

**Question:** Show that 0 is a Characteristic root of a matrix if and only if the matrix is singular.

**Solution:** Let A be any square matrix.

We now have,

‘O’ is a Characteristic root of A `0’ is a root of the equation | A-xI | =0

 | A-0.I | =0

 | A-0 | =0

 | A | =0

 A is singular.

**Question:** Prove that the Characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

**Solution:** Let  be any upper triangular matrix.

 The Characteristic equation of the matrix A is,

| A-xI | =0

= 0

()()………………..()=0

 x=, , ……………., .

 The Characteristic roots of the matrix A are , , ……………., , which are the diagonal elements of the matrix A.

Similarly, we can show that the Characteristic roots of the lower triangular matrix are also the diagonal elements of the matrix.

Hence, the Characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

**Question:** Find the eigenvalues and eigenvectors of the following matrices:

1.  (ii)  (iii)  (iv) 

**Solution (i):** Given that,



The characteristic equation of the matrix A is,

| A-xI |=0

 

(5-x)(2-x) – 4 = 0

x2 -7x+6= 0

(x-1)(x-6)= 0

 x= 1, 6

Thus the eigenvalues of the matrix A are 1, 6.

Let  be an eigenvector of the matrix A corresponding to the eigenvalue 1. Then,

AX1=1X1

 =1

=

=

 4x1=-4x2

 x1=-x2



Hence,  is an eigenvector of A corresponding to the eigenvalue 1. The set of all eigenvector of A corresponding to the eigenvalue 1 is given by c1X1, where c1 is any non-zero scalar.

Again, let  be an eigenvector of the matrix A corresponding to the eigenvalue 6. Then,

AX2=6X2

 =6

=

=

 -y1= -4x2

 y1= 4y2



Hence,  is an eigenvector of A corresponding to the eigenvalue 6. The set of all eigenvector of A corresponding to the eigenvalue 6 is given by c2X2, where c2 is any non-zero scalar.

**Solution (iii):** Given that,



The characteristic equation of the matrix A is given by,

| A-xI |=0

 = 0

(7-x){(-2-x)(-8-x)-0} - 0 – 3{ 0 – 18(-2-x)} = 0

(7-x)(2+x)(8+x) – 54(2+x) = 0

(2+x){(7-x)(8+x)} – 54 = 0

2+x=0 or (7-x)(8+x)- 54 = 0

x= -2 or 56+7x-8x-x2-54=0

 x= -2 or 2-x-x2=0

 x= -2 or x2 +x-2=0

 x= -2 or x2 +2x-x-2=0

 x= -2 or x(x+2) -1(x+2)=0

 x= -2 or (x+2)(x -1) =0

 x= -2 or x=-2 or x= 1

Thus the distinct eigenvalues of the matrix A are -2 and 1.

Let  be an eigenvector of the matrix A corresponding to the eigenvalue -2. Then,

AX1=-2X1

= -2

= 

= -2x1 ; =-2x2 and = -2x3

9x1=3x3 ; 9x1=3x3 and 18x1=6x3

3x1=x3



Thus we choose x1=1 and x3=3. But we have no information about x2. In fact , we have found that x2 can be chosen arbitrarily, and independent of x1 and x3. This allow us to choose two linearly independent eigenvectors corresponding to the eigenvalue x= -2, such as  and . Thus the set of all eigenvectors corresponding to the eigenvalue x= -2 can be written as linear combinations c1X1+c2X2, where c1 and c2 are non-zero scalars.

Again, let Let  be an eigenvector of the matrix A corresponding to the eigenvalue 1. Then,

AX3=X3

= 1

= 

= y1 ; = y2 and = y3

6y1=3y3 ; - 9y1 – 3y2 +3y3 =0 and 18y1=9y3

2y1=y3 ; - 9y1 – 3y2 +3y3 =0 and 2y1=y3

 Now,

 2y1=y3



Thus we take y1= 1 and y3= 2.

Again.

- 9y1 – 3y2 +3y3 =0

-9.1-3y2+3.2=0

-3 =3y2

y2=-1

 is an eigenvector of A corresponding to the eigenvalue 1. The set of all eigenvectors of A corresponding to the eigenvalue 1 is given by c3X3, where c3 is any non-zero scalar.

**Solution (iv):** Given that,



The characteristic equation of the matrix A is given by,

| A-xI |=0

 = 0

(1-x)(2-x)2  = 0

 x= 1, 2, 2

Thus the distinct eigenvalues of the matrix A are 1 and 2.

Let  be an eigenvector of the matrix A corresponding to the eigenvalue 1. Then,

AX1= 1X1

= 1.

= 

 = ; =  , and = 

-x3 =0, = 0 = 

 x3=0 = 0

Thus we have x2=0 and x3=0. But we have no information about x1. In fact , we have found that x1 can be chosen arbitrarily non-zero scalar, and independent of x2 and x3. Let us choose x1=1. Then  is an eigenvector of the matrix A corresponding to the eigenvalue 1. The set of all eigenvector of A corresponding to the eigenvalue 1 is given by c1X1, where c1 is any non-zero scalar.

Again , Let  be an eigenvector of the matrix A corresponding to the eigenvalue 2. Then,

AX2= 2X2

= 2.

= 

 = 2; =  , and = 2

y3= y3 y3= 0 2y2= y1

 

Thus we can take y1 = 2, y2 = 1, and y3 = 0.

 is an eigenvector of the matrix A corresponding to the eigenvalue 2. The set of all eigenvector of A corresponding to the eigenvalue 2 is given by c2X2, where c2 is any non-zero scalar.