05. Q. Show that the eigen value of $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ are ± 1 and the corresponding eigen vectors are $\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$ & $\begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$.

Solⁿ Given that, $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

The characteristic equⁿ of the given matrix is,

$$|A - xI| = 0$$

$$\Rightarrow \begin{vmatrix} \cos\theta - x & \sin\theta \\ \sin\theta & -\cos\theta - x \end{vmatrix} = 0$$

$$\Rightarrow -(\cos\theta - x)(\cos\theta + x) - \sin^2\theta = 0$$

$$\Rightarrow -(\cos^2\theta - x^2) - \sin^2\theta = 0$$

$$\Rightarrow -(\cos^2\theta + \sin^2\theta) + x^2 = 0$$

$$\Rightarrow x^2 - 1 = 0$$

$$\Rightarrow x = \pm 1$$

Hence, the eigen value of the given matrix are ± 1.

Let, $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value 1. Then

$$AX_1 = 1.X_1$$

$$\Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1\cos\theta + x_2\sin\theta \\ x_1\sin\theta - x_2\cos\theta \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\therefore x_1\cos\theta + x_2\sin\theta = x_1$$

$$\Rightarrow (1 - \cos\theta)x_1 = x_2\sin\theta$$

$$\Rightarrow \frac{x_1}{\sin\theta} = \frac{x_2}{(1 - \cos\theta)}$$

$$\Rightarrow \frac{x_1}{\cancel{2}\sin\theta/2\cos\theta/2} = \frac{x_2}{\cancel{2}\sin^2\theta/2}$$

$$\Rightarrow \frac{x_1}{\cos\theta/2} = \frac{x_2}{\sin\theta/2}$$

$\therefore X_1 = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$ is an eigen vector of the ~~vector~~ A corresponding the eigen value 1.

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Let $X_2 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be an eigen vector of the given matrix A corresponding to the eigen value -1.

$\therefore AX_2 = -1 . X_2$

$$\Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y_1 \cos\theta + y_2 \sin\theta \\ y_1 \sin\theta - y_2 \cos\theta \end{pmatrix} = \begin{pmatrix} -y_1 \\ -y_2 \end{pmatrix}$$

$$\therefore y_1 \cos\theta + y_2 \sin\theta = -y_1$$

$$\Rightarrow (\cos\theta + 1) y_1 = -y_2 \sin\theta$$

$$\Rightarrow \frac{y_1}{\sin\theta} = -\frac{y_2}{\cos\theta + 1}$$

$$\Rightarrow \frac{y_1}{\cancel{2}\sin\theta/2 \cancel{\cos\theta/2}} = -\frac{y_2}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\Rightarrow \frac{y_1}{\sin\theta/2} = \frac{y_2}{-\cos\theta/2}$$

$\therefore X_2 = \begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$ is an eigen vector of A corresponding to the eigen value -1.