Note:-

(i) $o(a) \leq o(G)$.

(ii) $a^n = e \Rightarrow o(a) \leq n$

(iii) $o(e) = 1$.

Note:- For any two non-empty subset H and K of a group 'G'; we define,

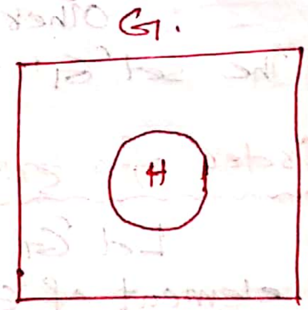
(i) $H \cup K = \{x \in G \mid x \in H \text{ or } x \in K\}$

(ii) $H \cap K = \{x \in G \mid x \in H \text{ and } x \in K\}$

(iii) $HK = \{hk \mid h \in H \text{ and } k \in K\}$

(iv) $KH = \{kh \mid k \in K \text{ and } h \in H\}$.

Sub-group :- A non empty subset 'H' of a group 'G' is said to be a subgroup of G if it is itself is a group with respect to the binary operation in G and it is denoted by $H \leq G$.



*****Theorem:-** A necessary and sufficient condition for a non-empty subset 'H' of a group "G" to be a subgroup is that $ab^{-1} \in H, \forall a, b \in H$.

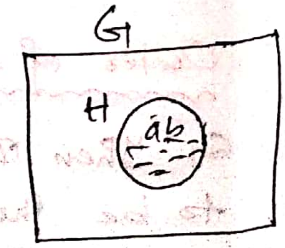
'OR'

A non-empty subset 'H' of a group 'G' to be a subgroup if and only if $ab^{-1} \in H, \forall a, b \in H$.

Proof:-> Given that 'H' is a non-empty subset of G. We are to show that

$H \leq G \Leftrightarrow ab^{-1} \in H, \forall a, b \in H$

1st let, $H \leq G$.



P.T.O.

Now, $a, b \in H$.

$\Rightarrow a, b^{-1} \in H$ [$\because H$ is itself is a group].

$\Rightarrow ab^{-1} \in H$ [by closer property]

$\therefore ab^{-1} \in H, \forall a, b \in H$

This shows that the condition is necessary.

Conversely, Let $ab^{-1} \in H \longrightarrow (1)$ $\forall a, b \in H$.

We now show that, 'H' is a sub-group of G.

(I) Since, $a \in H$

$\Rightarrow a, a \in H$

$\Rightarrow aa^{-1} \in H$ [using (1)]

$\Rightarrow e \in H$

This shows that 'H' ~~cont~~ contains the identity element.

(II) Again, $e, a \in H$

$\Rightarrow ea^{-1} \in H$ [using (1)]

$\Rightarrow a^{-1} \in H$.

This shows that every element of 'H' has its inverse.

(III) Since, $a, b \in H$.

$\Rightarrow a, b^{-1} \in H$ [by (II)]

$\Rightarrow a(b^{-1})^{-1} \in H$ [by (1)]

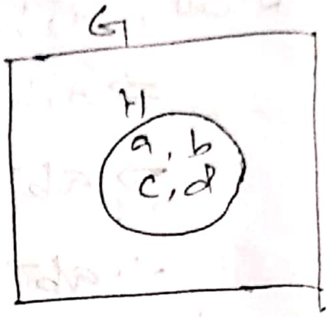
$\Rightarrow ab \in H., \forall a, b \in H$.

This shows that closure property hold good in H.//

(iv) Let, $a, b, c \in H$

$\Rightarrow a, b, c \in G \quad [H \subseteq G]$

$\Rightarrow a(bc) = (ab)c$



This shows that associative law hold good in H.

Hence, 'H' is a group and therefore 'H' is a sub-group of G i.e, $H \leq G$.

This shows that the condition is sufficient.

$\therefore H \leq G \Leftrightarrow ab^{-1} \in H, \ \forall a, b \in H$

Note:-

(i) $H \leq G \Leftrightarrow ab^{-1} \in H, \forall a, b \in H$

(ii) If 'G' is an additive group. Then $H \leq G \Leftrightarrow$

$a - b \in H, \forall a, b \in H.$