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2. Theorem :- If H is a subgroup of a group G, then

(i) $aH = bH \Leftrightarrow a^{-1}b \in H$, $\forall a, b \in G$.

(ii) $Ha = Hb \Leftrightarrow ab^{-1} \in H$, $\forall a, b \in G$

Proof :- given that $H \leq G$.

We now have,

$$aH = bH \Leftrightarrow a^{-1}(aH) = a^{-1}(bH)$$

$$\Leftrightarrow (a^{-1}a)H = (a^{-1}b)H$$

$$\Leftrightarrow eH = (a^{-1}b)H$$

$$\Leftrightarrow H = (a^{-1}b)H \quad [\because eH = H]$$

$$\Leftrightarrow (a^{-1}b)H = H$$

$$\Leftrightarrow a^{-1}b \in H \quad [\because xH = H \Leftrightarrow x \in H].$$

(ii) We have,

$$Ha = Hb \Leftrightarrow (Ha)b^{-1} = (Hb)b^{-1}$$

$$\Leftrightarrow H(ab^{-1}) = H(bb^{-1})$$

$$\Leftrightarrow H(ab^{-1}) = He$$

$$\Leftrightarrow H(ab^{-1}) = H \quad [\because He = H].$$

$$\Leftrightarrow ab^{-1} \in H \quad [Hx = H \Leftrightarrow x \in H].$$

$\therefore Ha = Hb \Leftrightarrow ab^{-1} \in H$ ///

3. Theorem :- If H is a subgroup of a group G, then prove that any two right coset of H in G either disjoint or equal.

'OR'

If H is a subgroup of a group G, Then prove that either $Ha = Hb$ or Ha & Hb are disjoint.

Proof :- Let Ha and Hb be any two right cosets of H in G.
We are to show that,

$$\text{either } Ha \cap Hb = \phi \text{ or } Ha = Hb.$$

Let $Ha \cap Hb \neq \phi$. To show $Ha = Hb$.

Let $x \in Ha \cap Hb$.

$\Rightarrow x \in Ha$ & $x \in Hb$.

$\Rightarrow x = ha$ & $x = h_1 b$, where $h, h_1 \in H$.

$\Rightarrow ha = h_1 b$

$\Rightarrow (ha)b^{-1} = (h_1 b)b^{-1}$

$\Rightarrow h(ab^{-1}) = h_1(bb^{-1})$

$\Rightarrow h(ab^{-1}) = h_1 e$

$\Rightarrow h(ab^{-1}) = h_1$

$\Rightarrow h^{-1}h(ab^{-1}) = h^{-1}h_1$

$\Rightarrow e(ab^{-1}) = h^{-1}h_1$

$\Rightarrow ab^{-1} = h^{-1}h_1$

$\Rightarrow ab^{-1} = h^{-1}h_1 \in H$ [$\because H \leq G$]

$\Rightarrow ab^{-1} \in H$

$\Rightarrow Ha = Hb$. [$\because Ha = Hb \Leftrightarrow ab^{-1} \in H$].

Similarly if $Ha \neq Hb$, then we can show that $Ha \cap Hb = \phi$.

'09' Theorem:- If H is a subgroup of a group G, then prove that either $aH = bH$ or aH & bH are disjoint.

4. Theorem:- If H is a sub-group of a group G, then prove that union of all distinct right (or left) coset of H in G is equal to the group G.

Proof:- Let R_c be the collection of all right co-sets of H in G.

i.e, $R_c = \{Ha \mid a \in G\}$.

We are to show that,

$$\bigcup_{a\in G} Ha = G.$$

Since every member of R_c is a subset of G,

$\therefore$ ~~$UHa \subseteq G$~~

$$\bigcup_{a\in G} Ha \subseteq G \quad \text{—(i)}$$

Again let $a \in G$

$\Rightarrow a = ea \in Ha \quad [\because e \in H]$

$\Rightarrow a \in Ha$

$\Rightarrow a \in \bigcup_{a\in G} Ha$.

$$\therefore G \subseteq \bigcup_{a\in G} Ha \quad \text{—(ii)}$$

$\therefore$ from (i) & (ii) we get—

$$G = \bigcup_{a\in G} Ha$$

Proved.

5. Theorem:- The number of distinct right cosets of H in G is equal to the no. of distinct left co-sets of H in G.

6. Theorem:- Any two right cosets or left cosets of H in G. Contains equal no. of distinct elements.

i.e, $o(Ha) = o(Hb)$ also $o(aH) = o(bH)$.

Note:- Since H is itself is a left coset as well as right coset, Therefore no. of distinct elements in any right or left coset of H in G is equal to that of H.

$\therefore o(H) = o(Ha) = o(Hb)$ etc.

& $o(H) = o(aH) = o(bH)$. etc