Date 13/12/09 *** Q:- State and prove Lagrange's Theorem on the order of a subgroup of a finite group.

Imp 18.

Soln:- Statement:- The order of a subgroup of a finite group is a factor (or divisor) of the order of the group.

Proof:- (1) Let G be a finite group of order n i.e. $o(G) = n$. and H be a subgroup of G. let $o(H) = m$.

We are to show that,

$$o(H) \mid o(G).$$

$$\Rightarrow m \mid n.$$

$$\Rightarrow n = mk, \text{ where } k \text{ is any +ve integers.}$$

Let R_c be the collection of all right co-sets of H in G.

ie, $R_c = \{Ha \mid a \in G\}$.

Since any two right co-sets either disjoint or equal therefore the members of R_c are mutually disjoint.

Since any two right co-sets contains equal no. of element therefore the number of distinct element in any member of R_c is equal to that of H.

Since G is a finite group therefore the no. of distinct right co-set of H in G must be finite.

Let $H, Ha_1, Ha_2, \ldots, Ha_{k-1}$ are the distinct right co-sets of H in G.

Since union of all distinct right co-sets of H in G. is equal to the group G,

$$\therefore G = H \cup Ha_1 \cup Ha_2 \cup \cdots \cup Ha_{k-1}.$$

$$\therefore o(G) = o(H) + o(Ha_1) + o(Ha_2) + \cdots + o(Ha_{k-1}).$$

$$\Rightarrow n = \underbrace{o(H) + o(H) + o(H) + \cdots + o(H)}_{k\text{-terms}}$$

$$\Rightarrow n = k \cdot o(H)$$

$$\Rightarrow n = k \cdot m$$

$$\Rightarrow n = km. \text{ Proved}$$

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Note:- (1) We have,

$$n = mk.$$

$$\Rightarrow \frac{n}{m} = k.$$

$$\Rightarrow \frac{n}{m} = \text{no. of distinct right co-sets of } H \text{ in } G.$$

$$\Rightarrow \frac{O(G)}{O(H)} = [G:H] \quad [\text{read it as } G \text{ index } H].$$

II. The order of an element of a finite group divise the order of the group.

$$\therefore O(a) \mid O(G), \ \forall a \in G.$$

Theorem:- Let G be a finite group of order n and $a \in G$. Then prove that $a^n = e$.

Proof:- Given that, G is a finite group of order n.
i.e, $O(G) = n$.

Let $O(a) = m$.

$$\Rightarrow a^m = e. \quad \text{---(i)}$$

Since order of an element divides the order of the group,

$$\therefore O(a) \mid O(G)$$

$$\Rightarrow m \mid n$$

$$\Rightarrow n = mk, \text{ where } k \in \mathbb{Z}^+ \quad \text{---(ii)}$$

Now, $a^n = a^{mk}$

$$= (a^m)^k$$

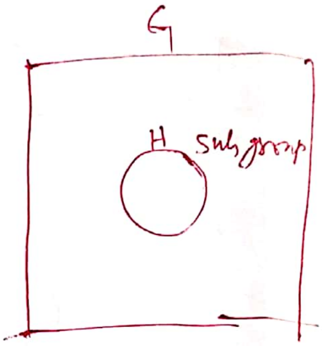
$$= e^k \ (\text{by (i)})$$

$$= e.$$

$\therefore a^n = e$ Proved

Normal subgroup :→

A subgroup H of a group G is said to be a normal subgroup of G if any left co-set of H in G is equal to its corresponding right co-sets, and it is denoted by $H \trianglelefteq G$.



$\therefore H \trianglelefteq G \Leftrightarrow xH = Hx, \ \forall x \in G.$

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Theorem :→ ~~Prove that~~ A subgroup H of a group G to be a normal subgroup of G if and only if $\cancel{ghg^{-1}}$ $xhx^{-1} \in H$, $\forall x \in G$ and $h \in H$.

Proof :→ Given that H is a subgroup of group G. We are to show that

$$H \trianglelefteq G \Leftrightarrow xhx^{-1} \in H, \forall x \in G \ \& \ h \in H.$$

First let, $H \trianglelefteq G$. Then,

$xH = Hx, \ \forall x \in G.$

$\Rightarrow xh = h_1 x$, where $h, h_1 \in H$

$\Rightarrow xhx^{-1} = h_1 x x^{-1}$

$\Rightarrow xhx^{-1} = h_1 e$

$\Rightarrow xhx^{-1} = h_1 \in H$

$\Rightarrow xhx^{-1} \in H$

$\Rightarrow xhx^{-1} \in H$

$xhx^{-1} \in H, \ \forall x \in G \ \& \ h \in H$

This shows that the condition is necessary.

Conversly, let $xhx^{-1} \in H, \ \forall x \in G \ \& \ h \in H$ —(1)

We now show that $H \trianglelefteq G$ ie we have to show that

~~Hx~~ $xH = Hx. \ \forall x \in G.$