Let, $y \in xH \Rightarrow y = xh$, where $h \in H$

$$\Rightarrow y = xhe$$

$$\Rightarrow y = xh(x^{-1}x)$$

$$\Rightarrow y = (xhx^{-1})x$$

$$\Rightarrow y = h_1 x, \text{ where } h_1 = xhx^{-1} \in H$$

$$\Rightarrow y = h_1 x \in Hx$$

$$\therefore xH \subseteq Hx \longleftarrow \text{(ii)}$$

Again let $z \in Hx \Rightarrow z = hx$, where $h \in H$.

$$= e(hx)$$

$$= (xx^{-1})(hx)$$

$$= x(x^{-1}hx)$$

$$= x\{x^{-1}h(x^{-1})^{-1}\}$$

$$= xh_1, \text{ where } h_1 = x^{-1}h(x^{-1})^{-1} \in H$$

$$= xh_1 \in xH.$$

$$\therefore Hx \subseteq xH \longleftarrow \text{(iii)}$$

$\therefore$ From (ii) & (iii) we get —

$$xH = Hx, \ \forall x \in G.$$

This shows that $H \trianglelefteq G$.

Q:- Prove that intersection of two normal subgroup of a group is again a normal subgroup.

Proof:- Let H and K be any two normal subgroup of a group G. We now show that $H \cap K$ is a normal subgroup of G.

Since H and K both are normal subgroups of G. Therefore H and K are subgroups of G and hence $H \cap K$ is also a subgroup of G. [$\because$ intersection of two subgroups is again a subgroup].

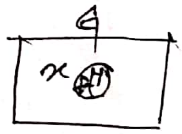
Let $x \in G$ & $h \in H \cap K$
$\Rightarrow h \in H$ & $h \in K$.

Since, $x \in G$, $h \in H$ and $H \trianglelefteq G \Rightarrow xhx^{-1} \in H$
Since, $x \in G$, $h \in K$ and $K \trianglelefteq G \Rightarrow xhx^{-1} \in K$.
$\therefore xhx^{-1} \in H \cap K$, $\forall x \in G$ & $h \in H \cap K$.

This shows that $H \cap K$ is a normal subgroup of G.

Theorem:- Every subgroup of an abelian group is normal.

Proof:- Let H be a subgroup of an abelian group G. We now show that $H \trianglelefteq G$.



Let $x \in G$ and $h \in H$.

Now, $xhx^{-1} = (xh)x^{-1}$
$= (hx)x^{-1}$ [$\because G$ is abelian, $\therefore xy = yx, \forall x, y \in G$]
$= h(xx^{-1})$ (by asso. law)
$= he$
$= h \in H$

$\therefore xhx^{-1} \in H$, $\forall x \in G$ & $h \in H$.

This shows that H is a normal subgroup of G.

Q:→ Let N be a normal subgroup of a group G and G/N be the set of all right co-sets of N in G i.e. $G/N = \{Nx \mid x \in G\}$. Show that G/N is a group w.r.to the following operation:

$$(Nx)(Ny) = Nxy, \ \forall x, y \in G.$$

Soln:→ Given that,

$$\cancel{G/N} \ G/N = \{Nx \mid x \in G\}$$

Define a binary operation on G by,

$$(Nx)(Ny) = Nxy, \ \forall \ x, y \in G.$$

We now show that G/N is a group.

(i) Let $Nx, Ny \in G/N$

$\Rightarrow x, y \in G$

$\Rightarrow xy \in G.$

$\Rightarrow Nxy \in G/N$

$\Rightarrow (Nx)(Ny) \in G/N, \ \forall Nx, Ny \in G/N$

(ii) Let $Nx, Ny, Nz \in G/N$

We now have,

$(Nx)\{(Ny)(Nz)\} = (Nx)(Nyz)$ (by defn).

$= Nx(yz)$ (by defn).

$= N(xy)z$ (by asso. law in G).

$= (Nxy)(Nz)$

$= \{(Nx)(Ny)\}(Nz).$

This shows that associative law hold good in G/N.