Ring :→

Defn :→ A non-empty set R together with two binary operation additive $(+)$ and multiplication $(\cdot)$ is said to be a ring if

R_1> $(R,+)$ an abelian group ie

(I) $x+y \in R, \forall x,y \in R$

(II) $x+(y+z) = (x+y)+z, \forall x,y,z \in R$

(III) $x+0 = 0+x = x, \forall x \in R$

(IV) for any $x \in R$, $\exists -x \in R$, s.t.

$$x+(-x) = (-x)+x = 0.$$

(V) $x+y = y+x, \forall x,y \in R$.

R_2> R satisfies associative law w.r.to multiplication on ie

$$x(y\cdot z) = (x\cdot y)\cdot z, \forall x,y,z \in R$$

R_3> Multiplication satisfies distributive law over addition

ie, (I) $x\cdot(y+z) = x\cdot y + x\cdot z$

(II) $(y+z)\cdot x = y\cdot x + z\cdot x$, $\forall x,y,z \in R$

Some Example of a Ring :→

Ex. The set of integer $\mathbb{Z}$, the set of real number $\mathbb{R}$ and The set of rational number $\mathbb{Q}$ is a ring.

Q:→ Define a ring. Show that the set K of all 2×2 matrices over reals is a ring under matrix addition and matrix multiplication. Is this ring commutative? Justify your answer.

Solⁿ:→ Definition:- See above.

2nd part:→ Here,

K = The set of all 2×2 matrices over real.

$$= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}.$$

We now show that K is a ring.

R_1) (i) We know that sum of two 2×2 matrices is again a 2×2 matrix.

Hence, $A+B \in K$, $\forall A, B \in K$.

(ii) Again matrix addition satisfies associative law:-

$\therefore A+(B+C) = (A+B)+C$, $\forall A, B, C \in K$

(iii) Clearly $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in K$, which is the additive identity of K.

(iv) For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$, we have $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in K$ which is the additive inverse of A.

(v) Again matrix addition satisfies commutative law:-

$\therefore A+B = B+A$, $\forall A, B \in K$

Hence, K is an additive abelian group.

R_2) We know that matrix multiplication is associative

$\therefore A(BC) = (AB) \cdot C$, $\forall A, B, C \in K$.

R_3) More over $A \cdot (B+C) = AB + AC$

& $(B+C)A = BA + CA$. $\forall A, B, C \in K$

Hence K is a ring.

3rd part :- R is not a commutative ring.

For,

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ & $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

$\therefore AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

$=$

$BA = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\therefore AB \neq BA$.