

Q.1: - Given an example of an integral domain which is not a field.

Soln: -> The set of integers  $\mathbb{Z}$  is an integral domain but it is not a field.

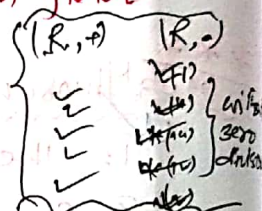
State <sup>\*\*\* ps</sup> by Theorem: -> Prove that a finite integral domain is a field.

Prove that a non-zero finite integral domain is a field.

Proof: -> Let  $R$  be a finite integral domain i.e.

$$\text{let } R = \{a_1, a_2, a_3, \dots, a_n\}$$

where  $a_1, a_2, \dots, a_n$  are distinct.



In order to show that  $R$  is a field, it is sufficient to show that  $R$  possesses multiplicative identity and every non-zero elt. of  $R$  has its multiplicative inverse.

Let  $a_i$  be any non-zero element of  $R$ .

We now consider the set,

$$R a_i = \{a_1 a_i, a_2 a_i, a_3 a_i, \dots, a_n a_i\}$$

Clearly all the elements of  $R a_i$  are distinct and  $R a_i \subseteq R$  and therefore

$$R a_i = R$$

$$\therefore a_i \in R$$

$$\Rightarrow a_i \in R a_i \quad [ \because R = R a_i ]$$

$$\Rightarrow a_i = a_j a_i, \text{ for some } 1 \leq j \leq n.$$

Let  $x$  be any element of  $R$ .

$$\therefore x \in R \Rightarrow x \in R a_i$$

$$\Rightarrow x = a_j a_i, \text{ for some } 1 \leq j \leq n$$

$$\text{Now, } x a_i = x (a_i a_i)$$

$$= (x a_i) a_i$$

$$\therefore x = x a_i \quad \text{--- (1) [by Right cancellation law]}$$

Since  $R$  is commutative,

$$\therefore x a_i = a_i x \quad \text{--- (11)}$$

∴ From (i) & (ii) we get —

$$x a_i = a_i x = x.$$

⇒  $a_i$  is the multiplicative identity of  $R$ .

Again,

$$a_i \in R$$

$$\Rightarrow a_i \in R a$$

$$\Rightarrow a_i = a_k a, \text{ for some } 1 \leq k \leq n.$$

$$\Rightarrow a_k a = a_i$$

⇒  $a_k$  is the inverse of  $a$ .

Thus every non-zero elt. of  $R$  has its multiplicative inverse.

Hence  $R$  is a field. #

Ex 3 → If  $R$  is a ring with unity and  $(xy)^n = x^n y^n, \forall x, y \in R$ .  
then prove that multiplication is commutative in  $R$ .

Proof → Given that,

$$(xy)^n = x^n y^n, \forall x, y \in R \quad \text{--- (i)}$$

∴  $x, y \in R$  and  $1 \in R$

$$\therefore (x(y+1))^n = x^n (y+1)^n \text{ (using (i)).}$$

$$\Rightarrow (xy + x)^n = x^n (y+1)^n$$

$$\Rightarrow (xy + x)(xy + x) = x^n (y+1)(y+1)$$

$$\Rightarrow (xy)^n + x y x + x x y + x^n = x^n (y^n + y + y + 1)$$

$$\Rightarrow x^n y^n + x y x + x x y + x^n = x^n y^n + x^n y + x^n y + x^n$$

$$\Rightarrow x y x = x^n y \quad \text{--- (ii)}$$

Replacing  $x$  by  $x+1$  in (ii) we get —

$$\Rightarrow (x+1) y (x+1) = (x+1)^n y$$

$$\Rightarrow (xy + y)(x+1) = (x+1)(x+1)y$$

$$\Rightarrow xy(x+1) + y(x+1) = (x+1)(xy + y)$$

$$\Rightarrow xyx + xy + yx + y = x^2 y + xy + xy + y$$

P.T.O.

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$$\Rightarrow xyx + yx = x^2y + xy$$

$$\Rightarrow yx + yx = x^2y + xy \quad (by (ii))$$

$$\Rightarrow yx = x^2y$$

$$\Rightarrow \forall x, y \in A, xy = yx$$

∴ multiplication is commutative in R

∴ R is a commutative ring.