Q:- Given an example of an integral domain which is not a field.

Soln:→ The set of integers $\mathbb{Z}$ is an integral domain but it is not a field.

Date 03/01/10 — *** 05, 07 **Theorem:→ Prove that a finite integral domain is a field.**

'or'

Prove that a non-zero finite integral domain is a field.

(R,+) (R,·)

Proof:→ Let R be a finite integral domain i.e.

let $R = \{a_1, a_2, a_3, \ldots, a_n\}$

where $a_1, a_2, \ldots, a_n$ are distinct.

In order to show that R is a field, it is sufficient to show that R possesses multiplicative identity and every non-zero elt. of R has its multiplicative inverse.

Let 'a' be any non-zero element of R.

We now consider the set,

$$Ra = \{a_1a, a_2a, a_3a, \ldots, a_na\}$$

clearly all the element of Ra are distinct and $Ra \subseteq R$ and therefore

$$Ra = R$$

$\therefore a \in R$

$\Rightarrow a \in Ra \quad [\because R = Ra]$

$\Rightarrow a = a_ia$, for some $1 \leq i \leq n$.

Let x be any element of R

$\therefore x \in R \Rightarrow x \in Ra$

$\Rightarrow x = a_ja$, for some $1 \leq j \leq n$

Now, $xa = x(a_ia)$

$= (xa_i)a$

$\therefore x = xa_i$ ——(i) [by Right cancellation law]

Since R is commutative,

$\therefore xa_i = a_ix$ ——(ii)

$\therefore$ From (i) & (ii) we get —

$x a_i = a_i x = x.$

$\Rightarrow a_i$ is the multiplicative identity of R

Again,

$a_i \in R$

$\Rightarrow a_i \in Ra$

$\Rightarrow a_i = a_k a$, for some $1 \leq k \leq n$.

$\Rightarrow a_k a = a_i$

$\Rightarrow a_k$ is the inverse of a.

Thus every non-zero elt. of R has its multiplicative inverse.

Hence R is a field. #

3) Q:→ If R is a ring with unity and $(xy)^2 = x^2y^2, \forall x, y \in R$. Then prove that multiplication is commutative in R

Proof:→ Given that,

$$(xy)^2 = x^2y^2, \forall x, y \in R \quad \text{—(i)}$$

$\because x, y \in R$ and $1 \in R$

$\therefore (x(y+1))^2 = x^2(y+1)^2$ (using (i)).

$\Rightarrow (xy+x)^2 = x^2(y+1)^2$

$\Rightarrow (xy+x)(xy+x) = x^2(y+1)(y+1)$

$\Rightarrow (xy)^2 + xyx + xxy + x^2 = x^2(y^2+y+y+1)$

$\Rightarrow x^2y^2 + xyx + x^2y + x^2 = x^2y^2 + x^2y + x^2y + x^2$

$\Rightarrow xyx = x^2y \quad \text{—(ii)}$

Replacing x by $x+1$ in (ii) we get —

$\Rightarrow (x+1)\,y\,(x+1) = (x+1)^2y$

$\Rightarrow (xy+y)(x+1) = (x+1)(x+1)y$

$\Rightarrow xy(x+1) + y(x+1) = (x+1)(xy+y)$

$\Rightarrow xyx + xy + yx + y = x^2y + xy + xy + y$

$\Rightarrow xyx + yx = x^2y + xy$

$\Rightarrow x^2y + yx = x^2y + xy$ (by (ii)),

$\Rightarrow yx = xy$

$\Rightarrow xy = yx, \forall x, y \in R$

$\therefore$ multiplication is commutative in R.

Hence R is a commutative ring.