V_3> Again for any $\alpha, \beta \in \mathbb{R}$, we have —

$$\{(\alpha+\beta)f\}(x) = (\alpha+\beta)f(x)$$
$$= (\alpha f(x) + \beta f(x))$$
$$= (\alpha f)x + (\beta f)(x)$$
$$= (\alpha f + \beta f)(x)$$
$$\therefore (\alpha+\beta)f = \alpha f + \beta f, \forall \alpha, \beta \in \mathbb{R}, \forall f \in X.$$

V_4> Also we have —

$$((\alpha\beta)f)(x) = (\alpha\beta)f(x)$$
$$= \alpha(\beta f(x))$$
$$= \alpha((\beta f)(x))$$
$$= (\alpha(\beta f))(x) \qquad [(\alpha f)x \Leftrightarrow \alpha(f(x))]$$
$$\therefore (\alpha\beta)f = \alpha(\beta f), \forall \alpha, \beta \in \mathbb{R}, \forall f \in X.$$

V_5> we have —

$$(1 \cdot f)(x) = 1 f(x)$$
$$= f(x)$$
$$\therefore 1f = f, \forall f \in X$$

where 1 is the multiplicative identity of the field $\mathbb{R}$.

Hence $X(\mathbb{R})$ is a vector space.

05. Q. Let M be the set of all $m \times n$ matrices with elements in a field F. Prove that M is a vector space over F under usual matrix addition and multiplecation of a matrix by an element of the field.

*

Sol^n: Given that, M is the set of all $m \times n$ matricies with elements of the field F. We now show that M(F) is a vetor space.

V_1) (i) We know that sum of two $m \times n$ matrices is again a $m \times n$ matrix.

$\therefore A+B \in M, \forall A, B \in M$

(ii) Again we know that matrix addition satisfies associative property.

$\therefore A+(B+C) = (A+B)+C, \forall A, B, C \in M.$

(iii) Clearly a zero matrix of size $m \times n$ is the addilive identity of M.

(iv) we know that every matrix has its addilive inverse. Thus every elements of M has its addibive inverse.

(v) Also we know that matrix addition satisfies commutative law.

$\therefore A+B = B+A, \forall A, B \in M.$

Hence, M is an additive abelian group.

More over, for any $\alpha, \beta \in F$ & $A, B, \in M$, we have —

V_2) $\alpha(A+B) = \alpha A + \alpha B$

V_3) $(\alpha+\beta)A = \alpha A + \beta A$

V_4) $(\alpha\beta)A = \alpha(\beta A)$

V_5) $1A = A$

Hence, M(F) is a vector space.

Dt. 6.9.08

Q. Let F be a field and $F[x]$ be the collection of all polynomials over F. Prove that $F[x]$ is a vector space.

Sol^n Let, $F[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\} \mid n \in N \ \& \ a_i \in F\}$

We now show that $F[x]$ is a vector space over F.

Let us define vector addition & scalar multiplication as —

(i) Vector addition :-

$$P(x) + Q(x) = (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + (b_0 + b_1x + b_2x^2 \cdots + b_mx^m)$$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_m)x^n \text{ if } n = m$$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_m)x^n \cdots + a_nx^n \quad \text{if } n > m$$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_m)x^n + b_mx^m \quad \text{if } m > n$$

(ii) Scalar multiplication :-

$$\alpha P(x) = \alpha(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$

$$= (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_n)x^n$$

where $\alpha \in F$ & $P(x) \in F[x]$

V_1) (i) From the defination it is clear that

$$P(x) + Q(x) \in F[x], \ \forall \ P(x), Q(x) \in F[x]$$

(ii) Again we have,

$$P(x) + \{Q(x) + R(x)\} = (a_0 + a_1x + \cdots + a_nx^n) + \{(b_0 + b_1x + \cdots + b_mx^m) + (c_0 + c_1x + \cdots + c_px^p)\}$$

$$= (a_0 + a_1x + \cdots + a_nx^n) + \{(b_0 + c_0) + (b_1 + c_1)x + \cdots + (b_m + c_p)x^m\} \text{ if } m = p$$

$$= \{a_0 + (b_0 + c_0)\} + \{a_1 + (b_1 + c_1)\}x + \cdots + \{a_n + (b_m + c_p)\}x^n \text{ if } m = n = p$$

$$= \{(a_0 + b_0) + c_0\} + \{(a_1 + b_1) + c_1\}x + \cdots + \{(a_n + b_m) + c_p\}x^n$$

$$= \{(a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_m)x^n\} + (c_0 + c_1x + \cdots + c_px^n)$$

P.T.O