Conversly, let W be a non-empty subset of $V(F)$ such that $\alpha x + \beta y \in W \rightarrow ①$ $\forall \alpha, \beta \in F$ & $x, y \in W$

We now show that W is a subspace of $V(F)$.

Taking $\alpha = 1$ & $\beta = -1$, we get from ①,

$$1.x + (-1)y \in W$$
$$\Rightarrow x - y \in W, \forall x, y \in W$$

This shows that W is a subgroup of an additive abelian group V and consequently W is an additive abelian group.

Again, since the element of W is also the element of V and $V(F)$ is a vector space,

$$\therefore \alpha(x+y) = \alpha x + \beta y, \forall x, y \in W \text{ \& } \forall \alpha \in F.$$
$$(\alpha+\beta)x = \alpha x + \beta x, \forall x \in W \text{ \& } \alpha, \beta \in F$$
$$(\alpha\beta)x = \alpha(\beta x), \forall x \in W, \forall \alpha, \beta \in F$$
$$\text{\& } 1.x = x, \forall x \in W,$$

where 1 is the multiplicative identity of F.

Hence, W is a vector space over F and therefore W is a subspace of $V(F)$.

This shows that the condition is sufficient.//

-o-and therefore W is a subspace of V(F).

01 **Theorem:-** Show that the intersection of any number of subspace of a vector space is a subspace. Give an example to show that union of two subspaces of a vector space is not a subspace.

Proof:- Let, $\{W_i \mid i \in \Delta$, where Δ is an index set$\}$ be any collection of vector subspaces of a vector space V(F). We are to show that $\bigcap_{i\in\Delta} W_i$ is also a subspace of V(F).

Let, $W = \bigcap_{i\in\Delta} W_i$

Let, $x, y \in W$ and $\alpha, \beta \in F$

Now, $x, y \in W \Rightarrow x, y \in \bigcap_{i\in\Delta} W_i$

$\Rightarrow x, y \in W_i, \ \forall i \in \Delta$

$\Rightarrow \alpha x + \beta y \in W_i, \ \forall i \in \Delta$.

[$\because W_i$'s are subspace of V(F)]

$\Rightarrow \alpha x + \beta y \in \bigcap_{i\in\Delta} W_i$

$\Rightarrow \alpha x + \beta y \in W$

$\therefore \alpha x + \beta y \in W, \ \forall \alpha, \beta \in F, \forall x, y \in W$

This shows that W is a subspace of V(F). //

P.T.O

2nd part

However union of two subspaces of a vector space may not be a subspace. for this we site an example bellow.

Let us consider the vector space $\mathbb{R}^2(\mathbb{R})$.

Let, $W_1 = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$

$W_2 = \{(0, y_1) \mid y_1 \in \mathbb{R}\}$

Clearly W_1 & W_2 both are subspace of $\mathbb{R}^2(\mathbb{R})$

However,

$$W_1 \cup W_2 = \{(x_1, 0), (0, y_1) \mid x_1, y_1 \in \mathbb{R}\}$$

is not a subspace of $\mathbb{R}^2(\mathbb{R})$.

For $(x_1, 0), (0, y_1) \in W_1 \cup W_2$, but

$$(x_1, 0) + (0, y_1) = (x_1, y_1) \notin W_1 \cup W_2$$

In particular $(1,0), (0,1) \in W_1 \cup W_2$, but

$$(1,0) + (0,1) = (1,1) \notin W_1 \cup W_2.$$

Theorem:- If W_1 & W_2 both are subspaces of $V(F)$, then $W_1 \cup W_2$ is also a subspace of $V(F)$ if an only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Proof:- Given that W_1 & W_2 both are subspaces of a vector space $V(F)$. We are to show that,

$W_1 \cup W_2$ is a subspace of $V(F) \Leftrightarrow$ either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

First let, $W_1 \cup W_2$ is a subspace of $V(F)$.

We are to show that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Let, $W_1 \nsubseteq W_2$, To show $W_2 \subseteq W_1$

Since, $W_1 \nsubseteq W_2 \Rightarrow \exists\, w_1 \in W_1$ s.t. $w_1 \notin W_2$

Let, x be any element of W_2

Now, $w_1 \in W_1$ & $x \in W_2$

$\Rightarrow w_1, x \in W_1 \cup W_2$

$\Rightarrow x + w_1 \in W_1 \cup W_2$ [$\because W_1 \cup W_2$ is a subspace]

$\Rightarrow x + w_1 \in W_1$ or $x + w_1 \in W_2$

P.T.O