But, $x+w_1 \notin W_2$, For if

$x+w_1 \in W_2$

$\Rightarrow x+w_1, x \in W_2$

$\Rightarrow (x+w_1)-x \in W_2$

$\Rightarrow w_1 \in W_2$, which contradics to the fact that $w_1 \notin W_2$

$\therefore x+w_1 \in W_1$

$\Rightarrow (x+w_1)-w_1 \in W_1$

$\Rightarrow x \in W_1$

$\therefore W_2 \subseteq W_1$

Slly, If $W_2 \notin W_1$, Then we can show that $W_1 \subseteq W_2$.

Conversly, Let either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

If $W_1 \subseteq W_2$ Then $W_1 \cup W_2 = W_2$ or $W_2 \subseteq W_1$,

then $W_1 \cup W_2 = W_1$. Since W_1 & W_2 both are subspace, therefore $W_1 \cup W_2$ is also a subspace of $V(F)$. //

Expiration of linearly independent and linearly dependent sets :-

A set of vectors $\{x_1, x_2, \ldots x_n\}$ is said to be linearly independent (L.I) set if for any $\alpha_1, \alpha_2, \ldots \alpha_n \in F$ such that -

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \bar{0}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Again, a set of vectors $\{x_1, x_2, \ldots x_n\}$ is said to be linear dependent (L.D) ~~set~~ if $\exists\ \alpha_1, \alpha_2, \ldots \alpha_n \in F$ not all zero (some may be zero) such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \bar{0}$$

Notes:- 1. A singleton set $\{\bar{0}\}$ is Linearly dependent.

2. A singleton set $\{u\}$, where $\vec{u} \neq 0$ is linearly Indep.

3. Any set containing $\bar{0}$ is linearly dependent.

4. Subset of linear ~~dependent~~ L.I set is linear independent.

5. Super set of a linear dependent set is linear dependent.

— o —

Theorem :- Prove that a subset of a linear independent set of a vector space is linear independent.

Proof :- Let, $S = \{x_1, x_2, \ldots, x_n\}$ be a linearly independent set of a vector space $V(F)$.

Let, $S_1 = \{x_1, x_2, \ldots, x_m\}$, where $m < n$.

clearly, $S_1 \subseteq S$.

We now show that S_1 is linear Independent.

Let, $\alpha_1, \alpha_2, \ldots \alpha_m \in F$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m = \bar{0}$$

$$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m + 0.x_{m+1} + 0.x_{m+2} + \cdots + 0.x_n = \bar{0}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \; [\because S \text{ is L.I.}]$$

This shows that the set S_1 is linear Independent.

* **Theorem:-** Prove that super set of a linearly dependent set of a vector space V(F) is linearly ~~in~~dependent.

Proof:- Let, $S=\{x_1, x_2, x_3, \dots x_n\}$ be a linearly dependent subset of vector space V(F).

Let, $S_1 = \{x_1, x_2, x_3, \dots x_n, y_1, y_2, \dots y_m\}$ be a super set of S. We now show that S_1 is linearly dependent.

Since, $x_1, x_2, \dots x_n$ are linearly dependent, therefore
$\exists\ \alpha_1, \alpha_2, \dots, \alpha_n \in F$ not all zero such that -

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \bar{0}$$

$$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + 0.y_1 + 0.y_2 + \dots + 0.y_m = \bar{0}$$

This shows that the set S_1 is linearly dependent.

Basis of a vector space:-

A set $S=\{x_1, x_2, \dots x_n\}$ is said to be a basis of the vector space V(F) if -

① S is linealy independent.

② $L(S) = V(F)$.

Dimention of a vector space:- The no. of distinct elemt. of a basis in a vector space V(F) is said to be the dimention of the vector space and it is denoted by dim.(V).