Dt. 22.9.08

Q. What do you mean by basic and dimension of a vector space? When a vector space is said to be finite dimensional? Given an example of an infinite dimensional vector space.

Sol^n

Q. Prove that the vectors $u=(1+i, 2i)$, $v=(1,1+i)$ is $\mathbb{C}^2$ are linearly dependent over the complex field $\mathbb{C}$ but are linearly independent over the real field $\mathbb{R}$.

Sol^n: Let, $S=\{u=(1+i,2i), v=(1,1+i)\}$

To show S is L.D. in $\mathbb{C}^2(\mathbb{C})$:-

Let, $\alpha_1 = i$, $\alpha_2=(1-i)$. Then clearly $\alpha_1, \alpha_2 \in \mathbb{C}$

We now have,

$$\alpha_1 u + \alpha_2 v = i(1+i, 2i) + (1-i)(1,1+i)$$
$$= (i-1, -2) + (1-i, (1+i)(1-i))$$
$$= (i-1, -2) + (1-i, 2)$$
$$= (0,0)$$

$\therefore \alpha_1 u + \alpha_2 v = (0,0)$ where $\alpha_1 \neq 0$, $\alpha_2 \neq 0$

This shows that S is linearly dependent in $\mathbb{C}^2(\mathbb{C})$.

To show S is linearly independent in $\mathbb{C}^2(\mathbb{R})$:-

Let, $\alpha, \beta \in \mathbb{R}$ such that

$\alpha u + \beta v = (0,0)$

$\Rightarrow \alpha(1+i, 2i) + \beta(1, 1+i) = (0,0)$

$\Rightarrow (\alpha+\alpha i, 2\alpha i) + (\beta, \beta+\beta i) = (0,0)$

$\Rightarrow (\alpha+\alpha i+\beta, 2\alpha i+\beta+\beta i) = (0,0)$

$\Rightarrow \underbrace{\alpha+\beta+\alpha i = 0}_{\to (1)}$ & $\beta+i(2\alpha+\beta) = 0 \to (11)$

$\Rightarrow (\alpha+\beta) + i\alpha = 0+i\cdot 0$ & $\beta + i(2\alpha+\beta) = 0+i\cdot 0$

$\Rightarrow \alpha = 0$ $\qquad\qquad \Rightarrow \beta = 0$

$\therefore \alpha u + \beta v = (0,0) \Rightarrow \alpha = 0, \beta = 0$

This shows that the set 'S' is lineary independent in $\mathbb{C}^2(\mathbb{R})$.

'08 Q. Prove that the set of vectors $\alpha_1 = (1,2,1)$ & $\alpha_2 = (2,1,0)$, $\alpha_3 = (1,-1,2)$ form a basis of $\mathbb{R}^3(\mathbb{R})$.

Sol^n Let, $S = \{\alpha_1 = (1,2,1), \alpha_2 = (2,1,0), \alpha_3 = (1,-1,2)\}$

To show that S is linearly indipendent :-

Let, $a, b, c \in \mathbb{R}$ such that

$a\alpha_1 + b\alpha_2 + c\alpha_3 = \bar{0}$

$\Rightarrow a(1,2,1) + b(2,1,0) + c(1,-1,2) = (0,0,0)$

$\Rightarrow (a, 2a, a) + (2b, b, 0) + (c, -c, 2c) = (0,0,0)$

$\Rightarrow (a+2b+c, 2a+b-c, a+2c) = (0,0,0)$

$\Rightarrow a+2b+c = 0 \longrightarrow (1)$

$2a+b-c = 0 \longrightarrow (2)$

$a+2c = 0 \longrightarrow (3)$

$(3) \Rightarrow a = -2c$

$\cancel{(1) \Rightarrow -2c+2b+c = 0}$

$\cancel{\Rightarrow 2b-c = 0}$

$(1)-(3) \Rightarrow 2b - c = 0.$

$\Rightarrow c = 2b \to (4)$

$(1)\times(2) \Rightarrow 2a+4b+2c = 0 \to (5)$

$(5)-(2) \Rightarrow 4b+2c - b + c = 0$

$\Rightarrow 3b + 3c = 0$

$\Rightarrow b + c = 0$

$\Rightarrow b+c=0$

$\Rightarrow b+2b=0$

$\Rightarrow 3b=0$

$\Rightarrow b=0$

$b=0$ putting in ①

$a+c=0 \rightarrow$ ⑥

③ - ⑥ $\Rightarrow c=0$

③ $\Rightarrow a=0$

$\therefore a=0$

$b=0$

$c=0$

$\therefore a\alpha_1+b\alpha_2+c\alpha_3=\bar{0} \Rightarrow a=b=c=0$

This shows that S is linearly independents.

To show $L(S)=\mathbb{R}^3(\mathbb{R})$:-

Let, $\alpha=(x_1,x_2,x_3)\in\mathbb{R}^3$, then $x_1,x_2,x_3\in\mathbb{R}$. If possible let, $a,b,c\in\mathbb{R}$ such that

$$\alpha=a\alpha_1+b\alpha_2+c\alpha_3$$

$$\Rightarrow (x_1,x_2,x_3)=a(1,2,1)+b(2,1,0)+c(1,-1,2)$$

$$\Rightarrow (x_1,x_2,x_3)=(a,2a,a)+(2b,b,0)+(c,-c,2c)$$

$$\Rightarrow (x_1,x_2,x_3)=(a+2b+c,\ 2a+b-c,\ a+2c)$$

$\therefore \cancel{x_1}\ a+2b+c=x_1 \longrightarrow$ ①

$2a+b-c=x_2 \longrightarrow$ ②

$a+2c=x_3 \longrightarrow$ ③

① - ② $\times 2 \Rightarrow -3a+3c=x_1-2x_2$

$\Rightarrow -3(x_3-2c)+3c=x_1-2x_2$ [by ③]

$\Rightarrow -3x_3+6c+3c=x_1-2x_2$

$\Rightarrow 9c=x_1-2x_2+3x_3$

$\Rightarrow c=\frac{1}{9}(x_1-2x_2+3x_3)$

$\therefore$ ③ $\Rightarrow a=x_3-\frac{2}{9}(x_1-2x_2+3x_3)$